

Geometric realization of γ -vectors of 2-truncated cubes.

V. D. Volodin*

Abstract

This paper continues investigation of the class of flag simple polytopes called 2-truncated cubes. It is an extended version of the short note [V3]. A 2-truncated cube is a polytope obtained from a cube by sequence of truncations of codimension 2 faces. Constructed uniquely defined function which maps any 2-truncated cube to a flag simplicial complex with f -vector equal to γ -vector of the polytope. As a corollary we obtain that γ -vectors of 2-truncated cubes satisfy Frankl-Furedi-Kalai inequalities.

1 Introduction

E.Nevo and T.K.Petersen (see [NP]) studied γ -vectors of generalized associahedra and proved that γ -vectors of Stasheff polytopes and Bott-Taubes polytopes can be realized as f -vectors of some simplicial complexes. This result gave rise to the following problem.

Problem (cf. [NP], Problem 6.4). *For given flag simple polytope P construct simplicial complex $\Delta(P)$ such that $\gamma(P) = f(\Delta(P))$.*

In [Ai1] N.Aisbett solved this problem for flag nestohedra. The construction introduced in [Ai1] used specific of building sets and was based on the fact that any flag nestohedron is a 2-truncated cube, i.e. can be obtained from the cube by sequence of truncations of codimension 2 faces (see [V1, V2]). Results about 2-truncated cubes one can find in [BV].

In the present paper we introduce the construction which for every 2-truncated cube gives required simplicial complex, i.e. solve the problem for class of all 2-truncated cubes. Moreover, we obtain that constructed complex is flag.

Theorem. *For every 2-truncated cube P^n there exists flag simplicial complex $\Delta(P)$ such that $\gamma(P) = f(\Delta(P))$.*

In the proof we use the construction that for a given sequence of truncations defines a unique simplicial complex with required f -vector. This construction is inductive and build such complexes (on same vertex set) for all faces of the 2-truncated cube. Then we obtain a function $\Delta(Q)$ on the set of faces G of P . This function is monotonic, i.e. $\Delta(Q_1) \subset \Delta(Q_2)$ provided by $Q_1 \subset Q_2$. As a corollary we prove that γ -vectors of 2-truncated cubes satisfy Frankl-Furedi-Kalai inequalities. For dimensions 2 and 3 the required complex is a set of $\gamma_1(P)$ points. For dimensions 4 and 5 the required complex is a graph with $\gamma_1(P)$ vertices and $\gamma_2(P)$ edges without triangles.

When this paper was in preparation there appeared [Ai2] in Archive. Central result of [Ai2] coincide with the central result of the note [V3] which is a short version of the present paper.

2 Face polynomials

The convex n -dimensional polytope P is called *simple* if its every vertex belongs to exactly n facets.

*This work is supported by the Russian Government project 11.G34.31.0053.

Let f_i be the number of i -dimensional faces of an n -dimensional polytope P . The vector (f_0, \dots, f_n) is called the f -vector of P . The F -polynomial of P is defined by:

$$F(P)(\alpha, t) = \alpha^n + f_{n-1}\alpha^{n-1}t + \dots + f_1\alpha t^{n-1} + f_0t^n.$$

The h -vector and H -polynomial of P are defined by:

$$H(P)(\alpha, t) = h_0\alpha^n + h_1\alpha^{n-1}t + \dots + h_{n-1}\alpha t^{n-1} + h_nt^n = F(P)(\alpha - t, t).$$

The g -vector of a simple polytope P is the vector $(g_0, g_1, \dots, g_{\lfloor \frac{n}{2} \rfloor})$, where $g_0 = 1$, $g_i = h_i - h_{i-1}$, $i > 0$.

The Dehn-Sommerville equations (see [Zi]) state that $H(P)$ is symmetric for any simple polytope. Therefore, it can be represented as a polynomial of $a = \alpha + t$ and $b = \alpha t$:

$$H(P) = \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} \gamma_i(\alpha t)^i (\alpha + t)^{n-2i}.$$

The γ -vector of P is the vector $(\gamma_0, \gamma_1, \dots, \gamma_{\lfloor \frac{n}{2} \rfloor})$. The γ -polynomial of P is defined by:

$$\gamma(P)(\tau) = \gamma_0 + \gamma_1\tau + \dots + \gamma_{\lfloor \frac{n}{2} \rfloor}\tau^{\lfloor \frac{n}{2} \rfloor}.$$

3 Class of 2-truncated cubes

In this section we introduce the class of 2-truncated cubes. Proofs of the propositions and more results about this class one can find in [BV].

Definition 1. We say that simple polytope \tilde{P} is obtained from simple polytope P by truncation of the face $G \subset P$, if simplicial complex $\partial\tilde{P}^*$ is obtained from the simplicial complex ∂P^* by stellar subdivision along the simplex σ_G corresponding to the face G . Polytope \tilde{P} has new facet corresponding to the new vertex $v_0 \in \partial\tilde{P}^*$.

Unformally, polytope \tilde{P} is obtained from P by shifting the support hyperplane of G inside polytope P . The new facet \tilde{F}_s of polytope \tilde{P} corresponding to the new vertex $v_0 \in \partial\tilde{P}^*$ is defined by the section. We will call it the *section facet* \tilde{F}_s .

Definition 2. Truncation of a face G of codimension 2 will be called 2-truncation. A combinatorial polytope obtained from a cube by 2-truncations will be called a 2-truncated cube.

Remark 1. In this case the section facet will have combinatorial type $G \times I$. After 2-truncation facet F of P either stays unchanged (if $G \supset F$ or $G \cap F = \emptyset$) or handles 2-truncation of face $F \cap G$. Then, for each face \tilde{Q} of \tilde{P} there exists a unique face Q such that either \tilde{Q} is obtained from Q by 2-truncation of $G \cap Q$ or \tilde{Q} is unchanged (or perturbed) face Q of P or $\tilde{Q} = Q \times I \subset G \times I$.

Proposition 1. Let the \tilde{P} be obtained from the simple polytope P by 2-truncation of the face G , then

$$\gamma(\tilde{P}) = \gamma(P) + \tau\gamma(G). \quad (1)$$

Proposition 2. Any 2-truncation keeps flagness.

Proposition 3. Every face of 2-truncated cube is a 2-truncated cube.

4 Main results

Simplicial complex is called *flag*, if its every clique forms a simplex. For simplicial complex K of dimension d the f -polynomial is defined by $f(K) := 1 + f_0t + \dots + f_d t^{d+1}$, where f_i are the numbers of i -dimensional faces. The central result of the paper is following.

Theorem 1. *For every 2-truncated cube P there exists a flag complex $\Delta(P)$ such that $\gamma(P) = f(\Delta(P))$.*

Let P be a 2-truncated cube with fixed sequence of truncations defined by section facets F_1, \dots, F_m . For every face $Q \subset P$ including $Q = P$, let us construct simplicial complex $\Delta(Q)$ on the vertex set $W(P) = \{w(F_1), \dots, w(F_m)\}$.

Construction 1. For $P = I^n$ we have $W(P) = \emptyset$ and $\Delta(Q) = \emptyset$ for all the faces.

Assume that required family of simplicial complexes is constructed for polytope P which is obtained from the cube by sequence of 2-truncations corresponding to sequence F_1, \dots, F_{m-1} of section facets of P . Let polytope \tilde{P} be obtained from P by 2-truncation of face $G_m \subset P$. Then, $W(\tilde{P}) = W(P) \cup \{w(F_m)\}$, where $w(F_m)$ corresponds to the new facet F_m of \tilde{P} .

Consider arbitrary face $\tilde{Q} \subset \tilde{P}$. Let Q be the face from remark 1. Then,

$$\Delta(\tilde{Q}) := \begin{cases} \Delta(Q) \cup (\Delta(G_m \cap Q) \star w(F_m)), & \text{if } \tilde{Q} \text{ is obtained from } Q \text{ by 2-truncation of } G_m \cap Q \subset Q; \\ \Delta(Q), & \text{otherwise.} \end{cases} \quad (2)$$

Remark 2. The number of connected components of $\Delta(P)$ is not greater than number of cubes among truncated faces G_1, \dots, G_m .

Lemma 1. *For every k -face Q^k of P we have*

$$\Delta(Q^k) = \bigcap_{F^{n-1} \supset Q^k} \Delta(F^{n-1})$$

Corollary 1. *Function $\Delta(\cdot)$ is monotonic, i.e. $\Delta(Q_1) \subset \Delta(Q_2)$ provided by $Q_1 \subset Q_2$.*

Proof of lemma 1. The lemma holds for $P = I^n$. Assume it holds for P and prove it for \tilde{P} obtained from P by 2-truncation of face G . Notice, that it is enough to prove lemma for faces of codimension 2. Let $\tilde{Q} = \tilde{F}_1 \cap \tilde{F}_2 \subset \tilde{P}$ be such a face. According to remark 1, we have 5 possible cases:

1. Both \tilde{F}_1 and \tilde{F}_2 are facets F_1 and F_2 of P not changed by truncation;
2. Facet \tilde{F}_1 is obtained from $F_1 \subset P$ by 2-truncation, facet \tilde{F}_2 is unchanged facet F_2 of P ;
3. Both faces \tilde{F}_1 and \tilde{F}_2 are obtained from faces F_1 and F_2 of P by 2-truncations;
4. Facet \tilde{F}_1 is the section facet \tilde{F}_s of \tilde{P} , facet \tilde{F}_2 is unchanged facet F_2 of P ;
5. Facet \tilde{F}_1 is the section facet \tilde{F}_s of \tilde{P} , facet \tilde{F}_2 is obtained from $F_2 \subset P$ by 2-truncation.

The case 1 is obvious. In the case 2 face \tilde{Q} is unchanged face Q of P . Then,

$$\begin{aligned} \Delta(\tilde{F}_1) \cap \Delta(\tilde{F}_2) &= (\Delta(F_1) \cup (\Delta(G \cap F_1) \star w(\tilde{F}_s))) \cap \Delta(F_2) = \\ &= \Delta(F_1) \cap \Delta(F_2) = \Delta(F_1 \cap F_2) = \Delta(\tilde{F}_1 \cap \tilde{F}_2) = \Delta(\tilde{Q}). \end{aligned}$$

In the case 3 face \tilde{Q} is obtained from face $Q = F_1 \cap F_2$ by truncation of its face $G \cap Q$. Then,

$$\begin{aligned} \Delta(\tilde{F}_1) \cap \Delta(\tilde{F}_2) &= (\Delta(F_1) \cup (\Delta(G \cap F_1) \star w(\tilde{F}_s))) \cap (\Delta(F_2) \cup (\Delta(G \cap F_2) \star w(\tilde{F}_s))) = \\ &= \Delta(F_1 \cap F_2) \cup (\Delta(G \cap F_1 \cap F_2) \star w(\tilde{F}_s)) = \Delta(\tilde{F}_1 \cap \tilde{F}_2) = \Delta(\tilde{Q}). \end{aligned}$$

In the case 4 we have $\Delta(\tilde{F}_s) = \Delta(G) \subset \Delta(F_2)$ since $G \subset F_2$. Then,

$$\Delta(\tilde{F}_1) \cap \Delta(\tilde{F}_2) = \Delta(G) \cap \Delta(F_2) = \Delta(G) = \Delta(\tilde{F}_1 \cap \tilde{F}_2) = \Delta(\tilde{Q}).$$

In the case 5 we have $\Delta(\tilde{F}_s \cap \tilde{F}_2) = \Delta(\tilde{F}_s \cap \tilde{F}_2 \cap \tilde{F}_3)$, where \tilde{F}_3 is a facet from the previous case. Then, the required relation follows from the previous cases and from the relation for polytope \tilde{F}_3 which holds by inductive assumption (by dimension). \square

Lemma 2. *For every face Q of P complex $\Delta(Q)$ is flag.*

Proof. On each step of construction 1 we merge two flag complexes $\Delta(P_{m-1})$ and $\Delta(G_m) \star w(F_m)$ with flag intersection $\Delta(G_m)$. Then, it is enough to prove that if $\Delta(P)$ contains some edge $\{v_1, v_2\}$ and for some its face Q complex $\Delta(Q)$ contains vertices v_1 and v_2 , then $\Delta(Q)$ contains also edge $\{v_1, v_2\}$.

Without loss of generality we assume that $v_1 \in \Delta(G_m)$ and $v_2 = w(F_m)$. Let \tilde{P} be obtained from P by 2-truncation of the face G_m and face \tilde{Q} be obtained from some face Q by 2-truncation of the face $G \cap Q$. We have $v_1 \in \Delta(G)$ and $v_1 \in \Delta(Q)$, then from lemma 1 follows that $v_1 \in \Delta(G \cap Q)$. Therefore, the edge $\{v_1, w(F_m)\}$ is contained in $\Delta((G \cap Q) \star w(F_m)) \subset \Delta(\tilde{Q})$. \square

Lemma 3. *For every face Q of P we have $\gamma(Q) = f(\Delta(Q))$.*

Proof. The lemma holds for $P = I^n$. From the formula 2 follows, that if the face \tilde{Q} is obtained from Q by 2-truncation, then $f(\Delta(\tilde{Q}))$ and $f(\Delta(Q))$ are connected by the next formula.

$$f(\Delta(\tilde{Q})) = f(\Delta(Q) + tf(\Delta(G \cap Q))).$$

Similar formula (1) connects γ -vectors of \tilde{Q} and Q . The lemma follows. \square

Theorem (Frankl-Furedi-Kalai, [FFR]). *Denote by $\binom{n}{k}_r$ the number of k -clique in Turan graph $T_{n,r}$. For natural numbers m, k and $r \geq k$ there exists unique canonical representation*

$$m = \binom{n_k}{k}_r + \dots + \binom{n_{k-s}}{k-s}_{r-s},$$

where $n_{k-i} - \lfloor \frac{n_{k-i}}{r-i} \rfloor > n_{k-i-1}$ for all $0 \leq i < s$ and $n_{k-s} \geq k-s > 0$. Denote

$$m^{\langle k \rangle_r} = \binom{n_k}{k+1}_r + \dots + \binom{n_{k-s}}{k-s+1}_{r-s}.$$

The integer vector (f_0, \dots, f_n) with nonnegative components is f -vector of some r -colorable simplicial complex K if and only if $f_k \leq f_k^{\langle k \rangle_r}$.

Then, using Frankl-Furedi-Kalai inequalities we obtain the following result which was proved for flag nestohedra in [Ai1].

Corollary 2. *Let P^n be a 2-truncated cube. Then $0 \leq \gamma_i \leq \gamma_k^{\langle k \rangle_r}$, where $k > 1, r = \lfloor \frac{n}{2} \rfloor$.*

Let us apply the obtained result to polytopes of dimensions 4 and 5. Their γ -vectors have only 3-components: $(1, \gamma_1, \gamma_2)$. In this case we obtain a graph with γ_1 vertices and γ_2 edges without triangles. Therefore, we have 3 inequalities:

1. $\gamma_1 \geq 0$;
2. $\gamma_2 \geq 0$;
3. $\gamma_2 \leq \frac{\gamma_1(\gamma_1-1)}{2}$.

References

- [Ai1] N. Aisbett, Frankl-Furedi-Kalai, Inequalities on the γ -vectors of flag nestohedra, arXiv:1203.4715v1.
- [Ai2] N. Aisbett, Frankl-Furedi-Kalai, gamma-vectors of edge subdivisions of the boundary of the cross polytope, arXiv:1209.1789.
- [BV] V.M.Buchstaber, V.D.Volodin, Combinatorial 2-truncated cubes and applications, Associahedra, Tamari Lattices, and Related Structures, Tamari Memorial Festschrift, Progress in Mathematics, Vol. 299, pp 161-186, 2012.
- [FFR] P. Frankl, Z. Furedi, and G. Kalai, Shadows of colored complexes, Math. Scand. 63 (1988) 169-178.
- [Fr] A. Frohmader, Face vectors of flag complexes, arXiv:math/0605673v1.

- [G] S. Gal, Real root conjecture fails for five- and higher-dimensional spheres, Discrete & Computational Geometry, vol. 34, no. 2, pp. 269-284, 2005; arXiv:math/0501046v1.
- [NP] E. Nevo, T. K. Petersen, On γ -vectors satisfying the Kruskal-Katona Inequalities, Discrete Comput. Geom, Vol. 45, 2010, pp. 503-521.
- [V1] V. Volodin, Cubical realizations of flag nestohedra and Gal's conjecture, arXiv:0912.5478v1.
- [V2] V. Volodin, Cubic realizations of flag nestohedra and proof of Gal's conjecture for them, Uspekhi Mat. Nauk, 65:1(391) (2010), 183-184.
- [V3] V. Volodin, Geometric realization of the γ -vectors of 2-truncated cube, Uspekhi Mat. Nauk, 67:3(405) (2012), 181-182.
- [Zi] G. Ziegler, Lectures on Polytopes, Springer-Verlag, 1995. (Graduate Texts in Math. V.152).

STEKLOV MATHEMATICAL INSTITUTE, MOSCOW, RUSSIA

DELONE LABORATORY OF DISCRETE AND COMPUTATIONAL GEOMETRY, YAROSLAVL STATE UNIVERSITY, YAROSLAVL, RUSSIA

E-mail adress: volodinvadim@gmail.com